

ON RELATIVE NORMAL MODES

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ABSTRACT. We generalize the Weinstein-Moser theorem on the existence of nonlinear normal modes near an equilibrium in a Hamiltonian system to a theorem on the existence of relative periodic orbits near a relative equilibrium in a Hamiltonian system with continuous symmetries. In particular we prove that under appropriate hypotheses there exist relative periodic orbits near relative equilibria even when these relative equilibria are singular points of the corresponding moment map, i.e. when the reduced spaces are singular.

Sur les oscillations normales relatives

RÉSUMÉ. On généralise le théorème d'existence de Weinstein-Moser des oscillations normales non-linéaires voisines d'un équilibre dans un système hamiltonien, à un théorème d'existence des orbites périodiques relatives voisines d'un équilibre relatif dans un système hamiltonien à symétrie continue. Entre autres on démontre, moyennant quelques hypothèses, qu'il existe des orbites périodiques relatives près des équilibres relatifs même lorsque ceux-ci sont des points singuliers de l'application de moment, c'est-à-dire lorsque les espaces réduits sont singuliers.

Version française abrégée.

Dans cette note on généralise le théorème de Weinstein-Moser [W1, Ms, W2, MnRS, B] sur les oscillations normales non-linéaires voisines d'un équilibre dans un système hamiltonien, à un théorème sur les orbites périodiques relatives voisines d'un équilibre relatif dans un système hamiltonien symétrique.

Soit M une variété symplectique munie d'une opération hamiltonienne d'un groupe de Lie connexe G et donc d'une application de moment $\Phi : M \rightarrow \mathfrak{g}^*$. Une orbite du champs hamiltonien X_h d'un hamiltonien G -invariant $h \in C^\infty(M)^G$ est un *équilibre relatif* [resp. une *orbite périodique relative*] si son image dans M/G est un point [resp. un lacet].

Les améliorations ultérieures du théorème de Weinstein-Moser se prêtent toutes à des généralisations par notre méthode; néanmoins, afin de ne pas alourdir l'exposé on traitera seulement la version originale:

Théorème de Weinstein. [W1] *Soit h un hamiltonien sur un vectoriel symplectique V avec $dh(0)$ nul et $d^2h(0)$ défini positif. Alors pour tout $\varepsilon > 0$ petit, $h^{-1}(h(0) + \varepsilon)$ porte au moins $\frac{1}{2} \dim V$ orbites périodiques de X_h .*

Soit $x \in M$ un point d'un équilibre relatif pour un hamiltonien G -invariant. Existe-t-il des orbites périodiques relatives voisines? A cela nulle difficulté si x est un point régulier de Φ , ou même dans le cas singulier si la dimension de la strate de son image \bar{x} dans l'espace réduit est strictement positive, car le problème se ramène alors au théorème de Weinstein sur cette strate, qui est stable par rapport à la dynamique de l'espace réduit stratifié [AMM, SL]. Mais que dire si la strate de \bar{x} est un point?

On verra qu'en l'occurrence il existe encore des familles d'orbites périodiques relatives voisines de l'équilibre relatif, pourvu qu'un certain 'remplaçant' du hessien soit défini en tant que forme

quadratique et que le groupe d'isotropie de $\Phi(x)$ soit un tore. On a en effet un théorème de correspondance que voici:

Théorème 1. *Soit M une variété symplectique munie d'une opération hamiltonienne d'un groupe de Lie connexe G et d'une application de moment $\Phi : M \rightarrow \mathfrak{g}^*$, et soit $h \in C^\infty(M)^G$ un hamiltonien G -invariant.*

Supposons que $x \in M$ appartienne à un équilibre relatif pour h et que le groupe d'isotropie de $\Phi(x)$ soit un tore. Alors il existe un vectoriel symplectique V muni d'une opération hamiltonienne du groupe d'isotropie G_x de x et un hamiltonien G_x -invariant $h_V \in C^\infty(V)^{G_x}$ tels que

1. *l'origine $0 \in V$ est un équilibre pour h_V ;*
2. *le hessien $d^2h_V(0)$ de h_V peut être calculé à partir de h ;*
3. *toute orbite périodique G_x -relative pour h_V dans V donne lieu à une famille d'orbites périodiques G -relatives pour h dans M .*

Ici V désigne le sous-espace symplectique maximal de $\ker d\Phi(x)$ (dit *tranche symplectique* en x) et $d^2h_V(0)$ est la forme quadratique induite sur V par $d^2(h - \langle \Phi | \xi \rangle)(x)|_{\ker d\Phi(x)}$, où ξ est une vitesse de x que l'on trouve dans

$$x \text{ appartient à un équilibre relatif pour } h \iff \exists \xi \in \mathfrak{g}, \quad d(h - \langle \Phi | \xi \rangle)(x) = 0.$$

La forme normale locale de Guillemin-Sternberg et de Marle [GS, Mr] permet de construire une fonction h_V sur V dont le hessien est $d^2h_V(0)$. Cette notion de hessien intervient dans l'étude de la stabilité et des bifurcations des équilibres relatifs aux points singuliers de l'application de moment [LS, O, OR].

Le Théorème 1, joint par exemple au théorème de Weinstein, conduit au résultat escompté:

Corollaire. *Soient $M, G, \Phi : M \rightarrow \mathfrak{g}^*, h \in C^\infty(M)^G$ comme dans le Théorème 1. Supposons que $x \in M$ appartienne à un équilibre relatif et que le groupe d'isotropie de $\Phi(x)$ soit un tore; appelons V la tranche symplectique en x . S'il existe une vitesse ξ de x telle que $d^2(h - \langle \Phi | \xi \rangle)(x)|_V$ est défini positif, alors pour tout $\varepsilon > 0$ petit, $\{y \in M \mid (h - \langle \Phi | \xi \rangle)(y) = (h - \langle \Phi | \xi \rangle)(x) + \varepsilon\}$ porte des familles d'orbites périodiques relatives.*

On peut sans doute se passer de l'hypothèse selon laquelle le groupe d'isotropie de $\Phi(x)$ est un tore, mais la démonstration ne promet guère d'être si simple. Signalons que dans le cas où G est compact, l'hypothèse est satisfaite de façon *générique*.

Esquisse de la démonstration.

D'abord, on se ramène au cas où G est un tore en passant à un sous-système hamiltonien dont le groupe de symétrie est le groupe d'isotropie de $\Phi(x)$, grâce à un théorème de Guillemin-Sternberg (voir [GLS], Corollaire 2.3.6). Ensuite, on plonge, par un symplectomorphisme équivariant, un voisinage de $G \cdot x$ dans $(T^*L \times V)/\Gamma$ (L est le tore complémentaire de la composante connexe K de 1 dans G_x , et $\Gamma = G_x/K$); en réduisant par L , on réduit ce dernier espace à V et h à h_V . Enfin, on vérifie que les orbites périodiques relatives pour h_V se relèvent à des orbites périodiques relatives pour h : les unes et les autres correspondent aux orbites périodiques du système obtenu soit en réduisant V par $\Gamma \times K$, soit en réduisant $(T^*L \times V)/\Gamma$ par $L \times K$.

Comme un exemple d'application, on remarque que le Corollaire assure l'existence des orbites périodiques relatives d'une paire de corps rigides à symétrie axiale liés par un potentiel qui dépend de l'angle entre les corps.

1. INTRODUCTION.

In this paper we generalize the Weinstein-Moser theorem ([W1, Ms, W2, MnRS, B] and references therein) on the nonlinear normal modes near an equilibrium in a Hamiltonian system to a theorem on the relative periodic orbits near a relative equilibrium in a symmetric Hamiltonian system.

Let M be a symplectic manifold, with a Hamiltonian action of a connected Lie group G and a moment map $\Phi : M \rightarrow \mathfrak{g}^*$. Recall that an orbit of the Hamiltonian vector field X_h of a G -invariant Hamiltonian $h \in C^\infty(M)^G$ is a *relative equilibrium* if its image in the orbit space M/G is a point, and that an orbit of X_h is a *relative periodic orbit* if its image in M/G is a closed curve.

Later improvements of the Weinstein-Moser theorem lend themselves to generalizations by our method; to streamline the presentation, however, we shall treat only the original version:

Weinstein's Theorem. [W1] *Let h be a Hamiltonian on a symplectic vector space V such that its differential at the origin $dh(0)$ is zero and its Hessian at the origin $d^2h(0)$ is positive definite. Then for every small $\varepsilon > 0$, the energy level $h^{-1}(h(0) + \varepsilon)$ carries at least $\frac{1}{2} \dim V$ periodic orbits of the Hamiltonian vector field of h .*

Now suppose $x \in M$ lies on a relative equilibrium for a G -invariant Hamiltonian h . If x is a *regular* point of the moment map Φ , then the reduced space at $\mu = \Phi(x)$ is smooth near the image \bar{x} of x . Provided appropriate conditions hold on the Hessian of the reduced Hamiltonian, Weinstein's theorem applied to the reduced system guarantees at least $\frac{1}{2} \dim(\text{reduced space})$ periodic orbits, and hence as many families of relative periodic orbits near the relative equilibrium. (If x lies on a relative periodic orbit, then the orbit through $g \cdot x$ is relative periodic also for every $g \in G$.) On the other hand, if x is a *singular* point of Φ , then the reduced space at μ is a stratified space, and the reduced dynamics preserves the stratification [AMM, SL]. Unless the stratum through \bar{x} is an isolated point, we have again $\frac{1}{2} \dim(\text{stratum})$ families of relative periodic orbits, provided appropriate conditions hold on the Hessian of the restriction of the reduced Hamiltonian to the stratum. But what if the stratum through \bar{x} is an isolated point? It is difficult to make sense of Hessians on singular spaces.

We shall show that in this case also there are families of relative periodic orbits near the relative equilibrium provided a certain substitute for the Hessian is definite and the isotropy group G_μ of μ is a torus. The proof amounts to a computation in 'good coordinates' that allows us to reduce our problem to Weinstein's theorem. We proceed via the following 'correspondence theorem':

Theorem 1. *Let M be a symplectic manifold, with a Hamiltonian action of a connected Lie group G and a moment map $\Phi : M \rightarrow \mathfrak{g}^*$, and let $h \in C^\infty(M)^G$ be a G -invariant Hamiltonian.*

Suppose $x \in M$ lies on a relative equilibrium for h and the isotropy group of $\Phi(x)$ is a torus. Then there exist a symplectic vector space V with a Hamiltonian action of the isotropy group G_x of x and a G_x -invariant Hamiltonian $h_V \in C^\infty(V)^{G_x}$ such that

1. *the origin $0 \in V$ is an equilibrium for h_V ;*
2. *the Hessian $d^2h_V(0)$ of h_V can be computed in terms of h ;*
3. *every G_x -relative periodic orbit for h_V on V sufficiently close to the origin gives rise to a family of G -relative periodic orbits for h on M .*

The meaning of the vector space V and of the quadratic form $d^2h_V(0)$ is as follows. Note that

$$x \in M \text{ lies on a relative equilibrium for } h \iff d(h - \langle \Phi | \xi \rangle)(x) = 0 \text{ for some } \xi \in \mathfrak{g}.$$

The vector ξ is called a *velocity* of x . Velocity is not unique: any two velocities of x differ by a vector in the isotropy Lie algebra \mathfrak{g}_x of x .

The function $h - \langle \Phi | \xi \rangle$ is constant along the orbit $G_\mu \cdot x$, where as above G_μ is the isotropy group of $\mu = \Phi(x)$. The form $d^2(h - \langle \Phi | \xi \rangle)(x)|_{\ker d\Phi(x)}$ is therefore always degenerate and descends to a well-defined form on the vector space $V := \ker d\Phi(x)/T_x(G_\mu \cdot x)$; alternatively we can think of V as the maximal symplectic subspace of $\ker d\Phi(x)$. V is called the *symplectic slice* at x . It follows from the local normal form theorem of Guillemin-Sternberg and Marle [GS, Mr] that there exists a G_x -invariant symplectic submanifold Σ passing through x such that $T_x\Sigma = V$. Thus, locally $\Sigma \simeq V$ as symplectic manifolds, with x corresponding to the origin in V . The function h_V in Theorem 1 is the restriction $(h - \langle \Phi | \xi \rangle)|_\Sigma = (h - \langle \Phi | \xi \rangle)|_V$. Since Hessians are functorial, $d^2h_V(0) = d^2(h - \langle \Phi | \xi \rangle)|_V(0) = d^2(h - \langle \Phi | \xi \rangle)(x)|_V$. This notion of Hessian has been used in [LS, O, OR] to study the stability and bifurcation of relative equilibria at singular points of the moment map.

As an example of applications of Theorem 1, we combine it with Weinstein's theorem to obtain:

Corollary. *Let $M, G, \Phi : M \rightarrow \mathfrak{g}^*$, $h \in C^\infty(M)^G$ be as in Theorem 1. Suppose $x \in M$ lies on a relative equilibrium for h and the isotropy group of $\Phi(x)$ is a torus; call V the symplectic slice at x . If there is a velocity ξ of x for which $d^2(h - \langle \Phi | \xi \rangle)(x)|_V$ is positive definite, then for every small $\varepsilon > 0$, the level set $\{y \in M \mid (h - \langle \Phi | \xi \rangle)(y) = (h - \langle \Phi | \xi \rangle)(x) + \varepsilon\}$ carries families of relative periodic orbits.*

We expect that the assumption on the isotropy group of $\Phi(x)$ being a torus can be dropped, but the proof is unlikely to be quite so simple. In case G is compact, the assumption is satisfied *generically*.

2. PROOF OF THEOREM 1.

Let $M, G, \Phi : M \rightarrow \mathfrak{g}^*$, $h \in C^\infty(M)^G$ be as in the statement of Theorem 1. We are supposing that $x \in M$ lies on a relative equilibrium for h and that the isotropy group G_μ of $\mu = \Phi(x)$ is a torus.

First, we show that it suffices to consider the case when the whole G is a torus. Since G_μ is compact, we can produce an $Ad^\dagger(G_\mu)$ -invariant inner product on \mathfrak{g}^* and take the corresponding G_μ -equivariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{h}$. Then a small enough G_μ -invariant neighborhood B of μ in the affine plane $\mu + \mathfrak{h}^\circ$ is transverse to the moment map (here \mathfrak{h}° denotes the annihilator of \mathfrak{h} in \mathfrak{g}). Hence $R := \Phi^{-1}(B)$ is a G_μ -invariant submanifold of M containing x . In fact, by the symplectic cross-section theorem of Guillemin and Sternberg (cf. [GLS], Corollary 2.3.6), R is a *symplectic* submanifold of M and the action of G_μ on R is Hamiltonian; the moment map for this action is the restriction of Φ to R followed by the natural projection $\mathfrak{g}^* \rightarrow \mathfrak{g}_\mu^*$. Since $\mathfrak{g}^* \rightarrow \mathfrak{g}_\mu^*$ restricted to $\mu + \mathfrak{h}^\circ$ is an isomorphism, the moment map for the action of G_μ on R is $\Phi|_R$ up to a linear isomorphism. It follows that

$$\ker d\Phi(y) = \ker d(\Phi|_R)(y)$$

for every $y \in R$ (cf. [LS], Lemma 2.5). Moreover, because h is G -invariant, the flow of X_h preserves the fibers of the moment map, and so the flow preserves R . It follows that

$$(X_h)|_R = X_{(h|_R)}$$

(cf. [L], p.218). Thus, we have found a Hamiltonian sub-system $(R, G_\mu, \Phi|_R, h|_R)$ for which the symmetry group G_μ is a torus. Passing to this sub-system, we may and shall assume without loss of generality that G is a torus and $G = G_\mu$.

Second, we construct the Hamiltonian h_V . The connected component K of 1 in the isotropy group G_x of x is a torus and has a complementary torus L , so that $G = L \times K$. The finite abelian group $\Gamma = G_x/K$ may be regarded as a subgroup of L . Then Γ acts symplectically (by

the lifted action) on $T^*L = L \times \mathfrak{l}^*$ and on the symplectic slice V at x . Hence Γ acts diagonally on $T^*L \times V$. Note that G too acts on $T^*L \times V$: L acts on its own cotangent bundle and K acts on V . Hence G acts in a Hamiltonian fashion on $(T^*L \times V)/\Gamma$. The orbit of $[1, 0, 0] \in (T^*L \times V)/\Gamma$ is isotropic and is isomorphic to $G/(\Gamma \times K) \simeq G \cdot x$. By the equivariant isotropic embedding theorem, there exist G -invariant neighborhoods U_0 of $[1, 0, 0]$ in $(T^*L \times V)/\Gamma$ and U of x in M and an equivariant symplectomorphism $\sigma : U_0 \rightarrow U$ such that $\sigma([1, 0, 0]) = x$ and the derivative $d\sigma([1, 0, 0])$ sends $V \subset T_{[1, 0, 0]}(T^*L \times V)/\Gamma$ to $V \subset \ker d\Phi(x) \subset T_x M$ by the identity map. Define $\tilde{h} = \sigma^*(h - \langle \Phi | \xi \rangle)$. Because G is assumed to be abelian, ξ is trivially in the center of \mathfrak{g} , and so \tilde{h} is G -invariant. At this juncture it is convenient to pretend that $U_0 = (T^*L \times V)/\Gamma$. By the invariance of $\tilde{h} \in C^\infty((T^*L \times V)/\Gamma)^{L \times K}$, $\tilde{h}([l, \lambda, v]) = \tilde{h}([1, \lambda, v])$ for all $(l, \lambda, v) \in L \times \mathfrak{l}^* \times V$. Now define $h_V(v) = \tilde{h}([1, 0, v])$. This $h_V \in C^\infty(V)^{\Gamma \times K}$ is the desired Hamiltonian: we have

$$dh_V(0) = d\tilde{h}([1, 0, 0])|_V = 0$$

and

$$d^2 h_V(0) = d^2 \tilde{h}([1, 0, 0])|_V = d^2 (\sigma^*(h - \langle \Phi | \xi \rangle))([1, 0, 0])|_V = \sigma^* (d^2(h - \langle \Phi | \xi \rangle)(x)|_V).$$

Third and last, we prove that relative periodic orbits for h_V give rise to relative periodic orbits for h . Consider the fully reduced system (P, h_P) obtained by reducing the system (V, h_V) by the action of $\Gamma \times K$. We can arrive at (P, h_P) also by reducing the system $((T^*L \times V)/\Gamma, \tilde{h})$ by the action of $G = L \times K$. Thus, relative periodic orbits for h_V correspond to periodic orbits for h_P , which in turn give rise to relative periodic orbits for \tilde{h} , or equivalently to relative periodic orbits for $h - \langle \Phi | \xi \rangle$. But relative periodic orbits for $h - \langle \Phi | \xi \rangle$ are relative periodic orbits for h . Indeed, writing φ_f^t for the Hamiltonian flow of f , we have $\varphi_{h - \langle \Phi | \xi \rangle}^t = \varphi_{-\langle \Phi | \xi \rangle}^t \circ \varphi_h^t$ by the G -invariance of h . Therefore, $\varphi_{h - \langle \Phi | \xi \rangle}^t(x) = g \cdot x$ for some $g \in G$ if and only if $\varphi_h^t(x) = g' \cdot x$ with $g' = \exp(-t\xi)g \in G$. This concludes the proof of Theorem 1.

3. CONCLUDING REMARKS

I. Corollary proves the existence of relative periodic orbits for a pair of axially symmetric rigid bodies subject to a coupling potential which depends on the angle between the bodies. We expect the theorem to prove the existence of relative periodic orbits for the motion of Riemann ellipsoids.

II. In Theorem 1, we compute $d^2 h_V(0)$ from $d^2(h - \langle \Phi | \xi \rangle)(x)|_{\ker d\Phi(x)}$ by ‘killing’ the direction of degeneracy. This computation needs only the *existence* of the ‘Darboux coordinates’ $\sigma : (T^*L \times V)/\Gamma \supset U_0 \rightarrow M$ (see the proof above), but by computing without the explicit form of σ , we lose some information. Alternatively we can compute the Taylor expansion of σ and translate the higher-order information on the jet of the original Hamiltonian h into the information on the jet of the model Hamiltonian h_V .

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